Development of Simplified Formula for Froude-Krylov Force of 6-DOFs Acting on Monohull Ship

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1. INTRODUCTION

In ship design, highly accurate estimation of ship motion in waves is demanded from various viewpoints, including the safety and riding comfort of the crew, wave loads for hull structural design, added resistance in waves in propulsive performance, etc. Rational estimation of ship motion in waves is now possible by seakeeping analysis tools such as the strip method or 3D panel method, and these tools have been provided for practical use at design work sites. On the other hand, there is also high demand for estimation of ship motion by a simple method which does not rely on numerical analysis. For example, in estimation of wave loads for evaluation of structural strength, performing wave load analyses for individual ships would be a significant obstacle in terms of the workload required in hull structural design. Therefore, CSR (Common Structural Rules) [1], which is a set of rules for steel ships, provides a method for estimating wave loads by simplified formulae using the main parameters of the ship. For the same reason, the intact stability criteria (International Code on Intact Stability; IS Code) [2] established by the IMO requires evaluation of safety based on the effective wave slope coefficient and damping force in rolling motion obtained by a simplified estimation method.

Generally speaking, a tradeoff relationship exists between the "simplicity" of simplified estimation methods and their "estimation accuracy and range of applicability." If a formula is developed by fitting to lots of the results of calculations, it is difficult to guarantee accuracy for targets that deviate from the used sample data. For example, the formulae for ship motion and acceleration provided in the current CSR [1] were derived by fitting to calculations for bulkers and oil tankers, and although the formulae are simple, they are not suitable to apply for untargeted ship types and sizes. Conversely, because the estimation formula for the effective wave slope coefficient provided in the IS Code [2] requires shape information for each transverse section of the hull, it is a strict method with high accuracy but lacks simplicity. In contrast to these two approaches, the authors believe that it is possible to satisfy both "simplicity" and "accuracy and applicability" by a process of identifying the dominant factors based on physical consideration, investigating their effects.

With this background, in the present research, the authors developed simplified formulae for the linear Froude-Krylov force based on a physical consideration to enable simple estimation of the ship motion in waves of a monohull ship of any arbitrary ship type and size. Although the work by Jensen et al. [3] is an example of past research for a similar purpose, that method was based on a formulation based on strip theory for a box-shaped ship with uniform dimensions of $L \times B \times d$, and the influence of the fineness of the ship geometry is considered by coefficient processing so as to fit several ships. In contrast, in the present research, we developed formulae that consider hull-form parameters of a ship such as the principal -particulars and fineness coefficients to enable application to all ship types from fine to blunt hull types. The estimation accuracy of the developed formulae was validated by calculation and comparison of the Froude-Krylov forces for various wave directions and wave lengths by a linear 3-dimensional seakeeping program using the actual hull-forms of 77 ships under 2 loading conditions (full load, ballast).

This paper is limited to the development of formulae for the Froude-Krylov force. However, because the Froude-Krylov force accounts for the main components of hydrodynamic forces that act on a ship, expressing those components by explicit formulae has a complete significance in itself. Its importance varies depending on the mode of motion, as the Froude-Krylov force is the principal component which becomes the leading term in the long wave length region [4] [5], while radiation and scattering hydrodynamic forces are also important in the wave length region where motion is large. In contrast to this, it is known that the Froude-Krylov force is particularly dominant for ship motion under roll and surge conditions. Where roll is concerned, because the scattering hydrodynamic force and the sway-induced radiation hydrodynamic force have a mutually-canceling effect,

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accurate estimation is possible by an equation of motion with one degree of freedom (DOF) using only the Froude-Krylov moment in the wave exciting force [6) 7)]. This concept has also been adopted in the IS Code [2)]. Surge can be estimated approximately from the Froude-Krylov force and hull weight because the fore and aft parts of ships are slender and elongated, and this calculation method has been adopted in many strip methods which do not consider the ship longitudinal component of the hull surface normal vector [8)]. The simplified formulae of the Froude-Krylov forces proposed in this paper are considered to be particularly effective for use in simplified estimations of these motions.

2. DEFINITIONS

2.1 Hull-Form Parameters Used in Formulae

Eight hull-form parameters are used in the formulae in this paper: the ship length L (length between perpendiculars, Lpp), breadth B, mean draft d, block coefficient C_b (=∇/LBd: where ∇ means displaced volume), waterplane area coefficient C_w (= Aw/LB: where Aw means the waterplane area), midship section area coefficient C_m, height of the center of gravity from keel KG and the longitudinal center of floatation using the center of gravity as the reference point x_f (= (longitudinal center of floatation LCF) – (longitudinal center of gravity LCG)). x_f is defined by Eq. (18) in the following. In this paper, the prismatic coefficient Cp (= C_b/C_m) and the vertical prismatic coefficient C_{vp} (= C_b/C_w) are used where appropriate. In addition, formulae were also developed for cases where the longitudinal metacentric height GM_L and the transverse metacentric height GM (defined by Eqs. (19) and (20) in the following) are used.

2.2 Coordinate System and Incident Wave

The definitions of the coordinate system and the directions of motions are shown in Fig. 1. The origins of the x, y, z coordinates are taken at the longitudinal center of gravity LCG, the centerline and the height of the waterline, respectively.

In this paper, the frequency response in regular waves was assumed based on linear theory and is expressed by the complex amplitude. That is, the amounts $a(t)$ of periodic variation are all handled by the complex number A defined by the following Eq. (1).

$$\begin{aligned} a(t) &= \Re[Ae^{i\omega_e t}] \\ &= \Re[A]\cos\omega_e t - \Im[A]\sin\omega_e t \\ &= |A|\cos(\omega_e t + \arg(A)) \end{aligned} \tag{1}$$

Where, ω_e is frequency of wave of encounter (frequency of encounter), and $\Re[A]$, $\Im[A]$, $|A|$, $\arg(A)$ are the real part, imaginary part, amplitude and argument of the complex number A, respectively.

The incident wave is defined as shown on the right in Fig. 1, and its velocity potential φ_0 is expressed as follows, assuming the instant when the crest of the wave reaches the position of the ship's center of gravity as the time reference ($t = 0$).

$$\phi_0 = \frac{ig\zeta_a}{\omega} e^{kz - ik(x\cos\beta + y\sin\beta)} \tag{2}$$

Where, g, ζ_a, ω, $k(=\omega^2/g)$, β are acceleration of gravity, and the wave amplitude, wave frequency, wave number and wave direction of the incident wave. In the following, the unit velocity potential shown below, which is nondimensionalized by $\omega/ig\zeta_a$, will be used.

$$\varphi_0 = e^{kz - ik(x\cos\beta + y\sin\beta)} \tag{3}$$

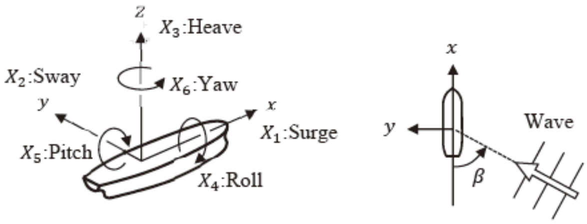


Figure 1 Definitions of coordinate system, motion and incident wave

2.3 Definition of Froude-Krylov Force and Asymptotic Value of Long Wave Length Region

The linear Froude-Krylov force is defined by the following equation as the integral of the velocity potential of the incident wave on the surface S_H of the ship's hull below the waterline.

$$E_i^{FK} = -\rho g \zeta_a \int_{S_H} \varphi_0 n_i \mathrm{d}S \qquad (i = 1\sim6) \tag{4}$$

$E_i^{FK} (i = 1 \text{ to } 6)$ are the Froude-Krylov forces in the surge, sway, heave, roll, pitch and yaw directions, respectively. When the basic flow field is approximated as a uniform flow, the definition in Eq. (4) holds independent of the ship's advance speed, and the influence of the advance speed is expressed only in the frequency of encounter ω_e. In Eq. (4), ρ is the density of seawater, and n_i $(i = 1 \text{ to } 6)$ represents the extension of the outward-facing unit normal vector $\{n_x, n_y, n_z\}^{\mathrm{T}}$ of the hull surface to 6 degrees of freedom (around the center of gravity) as defined by the following Eq. (5).

$$n_i = \begin{cases} n_x & (i = 1) \\ n_y & (i = 2) \\ n_z & (i = 3) \\ y n_z - (z - z_G) n_y & (i = 4) \\ (z - z_G) n_x - x n_z & (i = 5) \\ x n_y - y n_x & (i = 6) \end{cases} \tag{5}$$

Where, z_G is the z coordinate of the center of gravity (z_G = KG – d). In addition, the Froude-Krylov force in the surge, sway, heave and roll directions acting on a transverse section of a unit thickness (hereinafter referred to as "section Froude-Krylov force") is defined as shown by the following equation as the integral on the outer periphery $C_H(x)$ of the transverse section of the hull.

$$f_i^{FK}(x) = -\rho g \zeta_a \int_{C_H(x)} \varphi_0 n_i \mathrm{d}l \qquad (i = 1\sim4) \tag{6}$$

At this time, E_i^{FK} is expressed as follows using $f_i^{FK}(x)$.

$$E_i^{FK} = \begin{cases} \int_{x_A}^{x_F} f_i^{FK}(x) \mathrm{d}x & (i = 1\sim4) \\ \int_{x_A}^{x_F} -x f_3^{FK}(x) \mathrm{d}x & (i = 5) \\ \int_{x_A}^{x_F} x f_2^{FK}(x) \mathrm{d}x & (i = 6) \end{cases} \tag{7}$$

Where, x_A, x_F are the x coordinates of A.P. and F.P., respectively. In E_5^{FK}, E_6^{FK} of Eq. (7), the influence of the terms caused by n_x is considered to be negligibly small.

In the following, the Froude-Krylov force is nondimensionalized as follows, where the nondimensionalized quantity is indicated by an overbar.

$$\bar{E}_i^{FK} = \frac{E_i^{FK}}{\rho g \zeta_a L B \varepsilon_i} \quad (i = 1 \sim 6) \tag{8}$$

$$\bar{f}_i^{FK}(x) = \frac{f_i^{FK}(x)}{\rho g \zeta_a B \varepsilon_i} \quad (i = 2 \sim 4) \tag{9}$$

Here, ε_i is the representative length, which is defined as follows:

$$\varepsilon_i = \begin{cases} 1 & (i = 1 \sim 3) \\ B & (i = 4) \\ L & (i = 5,6) \end{cases} \tag{10}$$

Similarly, $\bar{x}$ obtained by nondimensionalizing x by L and $\bar{y}, \bar{z}$ obtained by nondimensionalizing y and z by B are used in the positional variables.

It is known that the asymptotic value of the Froude-Krylov force in the long wave length region corresponds to restoring force, and the consistency between the two influences the asymptotic value of motion [4]. Here, the exact value of the Froude-Krylov force will be presented in order to evaluate the asymptotic value in the long wave length region calculated by the simplified formulae. The following expressions are obtained by substituting the velocity potential of the incident wave shown in Eq. (3) into Eq. (4) and performing a Maclaurin expansion for k, and applying Gauss's divergence theorem to a scalar field (hereinafter referred to as the Gauss gradient theorem) .

$$\bar{E}_1^{FK} = i\bar{k}_l \frac{dC_b}{L} + O(k^2) \tag{11}$$

$$\bar{E}_3^{FK} = C_w - i\bar{k}_l \bar{x}_f C_w - kdC_b + O(k^2) \tag{12}$$

$$\bar{E}_5^{FK} = i\bar{k}_l \frac{dC_b}{L^2} \mathrm{GM_L} - \bar{x}_f C_w + O(k^2) \tag{13}$$

$$\bar{E}_2^{FK} = i\bar{k}_w \frac{dC_b}{B} + O(k^2) \tag{14}$$

$$\bar{E}_6^{FK} = \frac{\bar{k}_l \bar{k}_w}{L^3 B^2} \int_{V_H} (x^2 - y^2) \mathrm{d}V + O(k^3) \tag{15}$$

$$\bar{E}_4^{FK} = -i\bar{k}_w \frac{dC_b}{B^2} \mathrm{GM} + O(k^2) \tag{16}$$

Where, $O(k^n)$ is Landau's symbol, V_H is the displacement region and $\bar{k}_l, \bar{k}_w$ are the nondimensional wave numbers in the ship longitudinal and transverse directions, defined respectively as follows:

$$\bar{k}_l = kL\cos\beta, \bar{k}_w = kB\sin\beta \tag{17}$$

In the deformation of Eqs. (12) and (13), the following definition of the longitudinal center of floatation LCF is used.

$$\bar{x}_f = \frac{1}{C_w}\int_{\bar{x}_A}^{\bar{x}_F} \bar{x}\bar{B}_w(\bar{x})\mathrm{d}\bar{x} \tag{18}$$

Where, $\bar{B}_w(\bar{x})$ is a value obtained by dividing the waterline breadth $B_w(\bar{x})$ by B. $\mathrm{GM_L}$ on the right side of Eq. (13) is the longitudinal metacentric height (defined here as around the center of gravity) and GM on the right side of Eq. (16) is the transverse metacentric height. $\mathrm{GM_L}$ and GM are expressed as follows using the height of the center of buoyancy z_B and the height of the center of gravity z_G, respectively.

$$\mathrm{GM_L} = \underline{\frac{L^2}{dC_b}\int_{\bar{x}_A}^{\bar{x}_F} \bar{x}^2\bar{B}_w \mathrm{d}\bar{x}} + z_B - z_G \tag{19}$$

$$\mathrm{GM} = \underline{\frac{B^2}{dC_b}\int_{\bar{x}_A}^{\bar{x}_F} \frac{\{\bar{B}_w(\bar{x})\}^3}{12}\mathrm{d}\bar{x}} + z_B - z_G \tag{20}$$

The underlined parts on the right side of Eqs. (19) and (20) are the longitudinal metacentric radius $\mathrm{BM_L}$ and the transverse metacentric radius BM, respectively.

The correspondence between the asymptotic value of the Froude-Krylov force in the long wave length region and restoring force can be confirmed from Eqs. (12) to (16). The first term on the right side of Eq. (12) corresponds to the nondimensional restoring force coefficient of heave, the first term on the right side of Eq. (13), $\mathrm{GM_L} \times dC_b/L^2$, corresponds to the nondimensional restoring force coefficient of pitch, the second term on the right sides of Eqs. (12) and (13), $-\bar{x}_f C_w$, corresponds to the nondimensional restoring force coefficient of coupled heave-pitch motion and the first term on the right side of Eq. (16), $\mathrm{GM} \times dC_b/B^2$, corresponds to the nondimensional restoring force coefficient of roll.

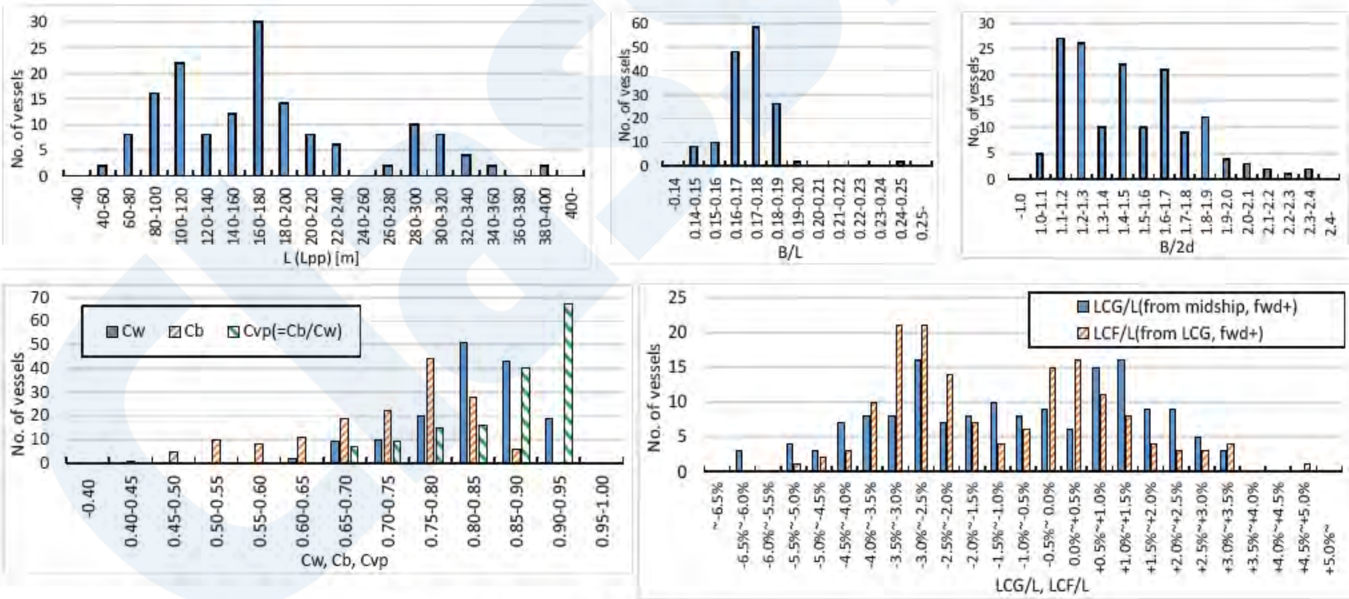


Figure 2 Histogram of hull-form parameters of target ships.

2.4 Numerical Calculation for Verification of Accuracy of Formulae

To verify the accuracy of the proposed formulae, calculations were performed by a linear 3-dimensional seakeeping program [9] developed by ClassNK, using 77 actually-existing ships under 2 loading conditions (full load and ballast condition). This program is based on a uniform flow approximation and calculates the Froude-Krylov force from the hull surface panels below the waterline by integral shown in Eq. (4). Because the targets here are general merchant ships, the area below the waterline is limited to bilateral symmetry and a monohull structure. However, a number of ship types were examined in this study, including bulk carriers, oil tankers, ore carriers, general cargo carriers, LNG carriers, LPG carriers, container ships, wood chip carriers, car carriers, RO-RO (roll-on/roll-off) ships, refrigerated cargo carriers (reefers) and cement carriers, and covered a wide range of L, C_w, C_b, B/L and $B/2d$, as shown in Fig. 2.

The wave conditions used in the comparison with the formulae in this paper included wave directions from β = 90° (beam sea) to 180° (head sea) in increments of 30°. For the wave length, wave length/ship length ratios λ/L of 0.5, 0.7, 1.0 and 1.5 were assumed. Where roll is concerned, estimation for a longer wave length region is important in some cases, but a numerical comparison was not carried out here because the asymptotic value for long wave lengths is evaluated separately by mathematical formulae. As mentioned above, the wave directions are limited to β = 90° to 180°. However, because the real part and imaginary part of the Froude-Krylov force acting on a bilaterally-symmetrical ship are symmetrical or antisymmetrial with respect to the wave direction, this range is neither excessive nor inadequate for verification of the real and imaginary parts. Here, it should also be noted that the Froude-Krylov force does not depend on the ship speed because the calculations are based on Eq. (4).

3. DEVELOPMENT OF SIMPLIFIED FORMULAE FOR FROUDE-KRYLOV FORCE

3.1 Basic Policy of Development

Because the Froude-Krylov force is the integral of the ship surface for a known scalar field, the key to the development of simplified formulae is "how to approximate the ship hull-form." Since the purpose of this research is to express the Froude-Krylov force by an elementary function in which the variables are limited to only the main ship parameters and wave conditions, ship hull-form is approximated by a function that can be integrated analytically so that it is determined uniquely by the main ship parameters. As described detailly in the following sections, different hull-forms were selected for each mode of motion so that the formulae are simple and rational as the evaluation of the integrated value. In particular, the hull-form is decided with care so that the asymptotic value in the long wave length region either coincides with or is a good approximation of the result given by the exact equation shown in section 2.3. Furthermore, for the ship surface integral, the section Froude-Krylov force $\bar{f}_i^{FK}(\bar{x})$ is defined and is then integrated in the ship longitudinal direction, and the integrand is simplified appropriately in this process. For example, the Smith correction factor ($e^{-kd'(\bar{x})}$; where $d'(\bar{x})$ is the section draft) appears in the section Froude-Krylov force, but because integration is difficult or impossible when treating its longitudinal distribution, the integrand is simplified by replacing the section draft $d'(\bar{x})$ with the constant d_e (hereinafter referred to as "equivalent draft") so that the integrals are equivalent.

3.2 Surge

As the point of departure of simplified methods for calculating $\bar{E}_1^{FK}$, the following expression, in which the Gauss gradient theorem is applied to Eq. (4), is often used.

$$\begin{aligned}\bar{E}_1^{FK} &= -\frac{1}{LB}\int_{V_H}\frac{\partial\varphi_0}{\partial x}\mathrm{d}V \\ &= i\bar{k}_l\int_{\bar{x}_A}^{\bar{x}_F}e^{-i\bar{k}_l\bar{x}}\left\{\int_{A_H(\bar{x})}e^{kz-iky\sin\beta}\frac{\mathrm{dydz}}{LB}\right\}\mathrm{d}\bar{x}\end{aligned} \tag{21}$$

Where, $A_H(\bar{x})$ is the transverse section below the waterline. $\bar{E}_1^{FK}$ is normally calculated based on Eq. (21) in strip method programs, which do not use n_x in calculations of hydrodynamic forces. The integral on $A_H(\bar{x})$ is solved by direct integration, or solved more simply by selecting a represented point of the wave particle velocity of the incident wave [8) 10)]. Here, $\bar{E}_1^{FK}$ is obtained analytically after approximating the section as a rectangle. Assuming the section geometry as rectangle with breadth $B'(\bar{x})$ and depth $d'(\bar{x})$, the integral on $A_H(\bar{x})$ can be expressed as follows:

$$\int_{A_H(\bar{x})}e^{kz-iky\sin\beta}\frac{\mathrm{dydz}}{LB} = \frac{1-e^{-kd'(\bar{x})}}{kL}\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w\bar{B}'(\bar{x})}{2} \tag{22}$$

Where, $\bar{B}'(\bar{x}) = B'(\bar{x})/B$. Substituting Eq. (22) into Eq. (21), $\bar{E}_1^{FK}$ is approximated as shown in Eq. (23).

$$\bar{E}_1^{FK} \cong i(1-e^{-kd_e})\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\frac{\bar{k}_l}{kL}\int_{\bar{x}_A}^{\bar{x}_F} e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} \tag{23}$$

In order to simplify the integral in the approximation in Eq. (23), the following approximation is assumed:

$$\sin\frac{\bar{k}_w\bar{B}'(\bar{x})}{2} \cong \bar{B}'(\bar{x})\sin\frac{\bar{k}_w}{2} \tag{24}$$

Furthermore, the draft $d'(x)$ in the Smith correction factor $e^{-kd'}$ is replaced with the equivalent draft d_e and removed from the integrand. Here, since it is considered possible to approximate the projected plane of the ship's shape below the waterline on the y-z plane as the rectangle $B \times dC_m$, it is assumed that $d_e = dC_m$. The distribution of $\bar{B}'(\bar{x})$ is assumed by a trapezoidal distribution of an area C_p with symmetry in the longitudinal direction, centering on the longitudinal center of buoyancy LCB.

$$\bar{B}'(\bar{x}) = \begin{cases} 1 & \text{for} \quad |\bar{x}| \le C_p - 0.5 \\ \dfrac{0.5-|\bar{x}|}{1-C_p} & \text{for} \quad C_p - 0.5 < |\bar{x}| \le 0.5 \end{cases} \tag{25}$$

The area of $\bar{B}'(\bar{x})$ is set to C_p in order to the nondimensional displacement is correspond to C_b (= C_pC_m). Substituting the above into Eq. (23), the following proposed formula is obtained.

$$\bar{E}_1^{FK} = i(1-e^{-kdC_m})\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{kL}\sin\frac{C_p\bar{k}_l}{2}\right)\left\{\frac{2}{(1-C_p)\bar{k}_l}\sin\frac{(1-C_p)\bar{k}_l}{2}\right\} \tag{26}$$

If the proposed formula shown in Eq. (26) is expanded for k, agreement of the asymptotic value in its long wave length region with the exact value given by Eq. (11) can be confirmed.

Figure 3 shows the comparison of $\bar{E}_1^{FK}$ by the developed formula shown in Eq. (26) and the numerical calculations for the actual ships shown in section 2.4. From Fig. 3, it can be understood that $\Im[\bar{E}_1^{FK}]$ has satisfactory accuracy for all ships and wave conditions. Regarding $\Re[\bar{E}_1^{FK}]$, in the calculations, this value becomes 0 by symmetric domain integration of odd functions because a anterior-posterior symmetric hull-form was assumed. In comparison with this, the value for the actual ships is at most about $\Re[\bar{E}_1^{FK}] = 0.02$, confirming that the influence of the anterior-posterior asymmetry of the hull-form can be neglected.

Since the midship section area coefficient of almost all general merchant ships is in the range of $C_m > 0.96$, there is virtually no reduction in accuracy in many cases even if $C_m = 1$ is assumed in the calculation. However, if $\bar{B}'(\bar{x})$ is not considered as trapezoidal as in Eq. (25), but is approximated by a rectangular distribution of the area C_p, $\bar{E}_1^{FK}$ is represented by an equation which does not contain the expression shown in the curly brackets ({}) on the right side of Eq. (26), and in this case, estimation accuracy decreased remarkably in the short wave length region. Because n_x has a value mainly in the vicinity of the ship bow and stern, the importance of the approximation of the shapes of the bow and stern is higher than that of other hydrodynamic forces. Therefore, a highly accurate formula was obtained by assuming that the distribution of breadth in the ship longitudinal direction is a trapezoid close to that of the actual hull-form (see Table 1).

3.3 Heave and Pitch

When the transverse sectional geometry of the hull is considered to be a rectangle with breadth $B'(\bar{x})$ and depth $d'(\bar{x})$, the section Froude-Krylov force in the z direction $\bar{f}_3^{FK}(\bar{x})$ is expressed by the following Eq. (27):

$$\bar{f}_3^{FK}(\bar{x}) = \left\{\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w\bar{B}'(\bar{x})}{2}\right\}e^{-kd'(\bar{x})-i\bar{k}_l\bar{x}} \tag{27}$$

By substituting Eq. (27) into Eq. (7), performing an approximation of Eq. (24), and removing the Smith correction factor from the integrant by using the equivalent draft d_e, $\bar{E}_3^{FK}$ and $\bar{E}_5^{FK}$ can be expressed as follows:

$$\bar{E}_3^{FK} \cong e^{-kd_e}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F} e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} \tag{28}$$

$$\bar{E}_5^{FK} \cong e^{-kd_e}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F} -\bar{x}e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} \tag{29}$$

Here, the equivalent draft d_e is assumed as average draft, i.e. $d_e = dC_{vp}$. Since $\bar{E}_3^{FK}$ and $\bar{E}_5^{FK}$ are integrals with respect to n_z, it is inferred that they are deeply related the shape of the projection plane of the hull-form in the z direction, that is, the shape of the waterline plane. Based on this idea, $\bar{B}'(\bar{x})$ is considered to be equivalent to the waterline breadth $\bar{B}_w(\bar{x})$. Assuming a rectangular distribution of the area C_w with its center at LCG,

$$\bar{B}'(\bar{x}) = \begin{cases} 1 & \text{for} \quad |\bar{x}-\bar{x}_f| \le C_w/2 \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

In case the above is used, the integrals of Eqs. (28) and (29) are as follows:

$$\int_{\bar{x}_A}^{\bar{x}_F} e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} = e^{-i\bar{k}_l\bar{x}_f}\left(\frac{2}{\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}\right) \tag{31}$$

$$\int_{\bar{x}_A}^{\bar{x}_F} -\bar{x}e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} = ie^{-i\bar{k}_l\bar{x}_f}\frac{1}{\bar{k}_l}\left\{\left(\frac{2}{\bar{k}_l}+2i\bar{x}_f\right)\sin\frac{C_w\bar{k}_l}{2} - C_w\cos\frac{C_w\bar{k}_l}{2}\right\} \tag{32}$$

The expression $e^{-i\bar{k}_l\bar{x}_f}$ on the right side of Eqs. (31) and (32) is the phase difference due to the fact that the center of action of the Froude-Krylov force is the LCF, while the reference phase of the incident wave is defined by the center gravity of the ship's hull. On the other hand, the $\bar{x}_f$ in the parentheses that can be seen on the right side of Eq. (32) is the lever of the center of action of the Froude-Krylov force and center of gravity owing to the fact that $\bar{E}_5^{FK}$ is defined as the moment around the center of gravity.

The above-mentioned equations are derived as a result of regarding the hull as a "box-shaped vessel with dimensions of $LC_w \times B \times dC_{vp}$ with its center at LCF." This approximation seems reasonable in the case of beam sea because the incident wave profile is uniform in the ship's longitudinal direction, i.e., $e^{-i\bar{k}_l\bar{x}} = 1$. However, in the case of head sea or following sea, the wave profile $e^{-i\bar{k}_l\bar{x}}$ changes in the longitudinal direction in the short wave length region and it is not reasonable to approximate the hull-form as a box-shape. In order to water plane in the short wave length region of longitudinal waves and the influence of the fineness of the ship under waterline without sacrificing the simplicity of the formula, $\bar{k}_l$ in the equation is replaced with the following $\bar{k}_l'$:

$$\bar{k}_l' = C_b^{-0.15}\bar{k}_l \tag{33}$$

This correction was applied to the ship longitudinal nondimensional wave number $\bar{k}_l$ to change the value under a condition of longitudinal waves in the short wave length region, and C_b was used in the correction to correct both fineness at the water plane (C_w) and fineness below the water plane (C_{vp}) by $Cb = C_wC_{vp}$. The exponent –0.15 was decided to obtain high agreement, based on the results for the actual ships.

From the foregoing discussion, the following equations are proposed as the simplified formulae of the Froude-Krylov forces for heave and pitch.

$$\bar{E}_3^{FK} = e^{-i\bar{k}_l\bar{x}_f - kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}\right) \tag{34}$$

$$\bar{E}_5^{FK} = ie^{-i\bar{k}_l\bar{x}_f - kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\frac{1}{\bar{k}_l'}\left\{\left(\frac{2}{\bar{k}_l'} + 2i\bar{x}_f\right)\sin\frac{C_w\bar{k}_l'}{2} - C_w\cos\frac{C_w\bar{k}_l'}{2}\right\} \tag{35}$$

In these formulae, $\bar{k}_l$ is used instead of $\bar{k}_l'$ is used in $e^{-i\bar{k}_l\bar{x}_f}$. This is based on the consideration that $\bar{E}_3^{FK}$ should achieve its maximum (or minimum) value at the instant when the crest (or trough) of the incident wave reaches the position of the LCF.

The comparison of the results of the developed formulae shown in (34) and (35) and the values obtained by the numerical calculations are shown in Fig. 4 and Fig. 5, and confirm that the formulae have satisfactory practical accuracy for all ship types and wave conditions. A good correlation can also be seen for $\Im[\bar{E}_3^{FK}]$ and $\Re[\bar{E}_5^{FK}]$, which are caused by the anterior-posterior asymmetry of the hull-form. This means that it is appropriate to regard the center of action of Froude-Krylov force in the z direction as being located at the LCF.

Regarding the amplitude in Eqs. (34) and (35), because Fig. 2 showed that the value of $\bar{x}_f$ is small, being about ±0.05, the terms for the squares of $\bar{x}_f$ are neglected, and the expressions are rewritten as shown below.

$$|\bar{E}_3^{FK}| = e^{-kdC_{vp}}\left|\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right|\left|\frac{2}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}\right| \tag{36}$$

$$|\bar{E}_5^{FK}| = e^{-kdC_{vp}}\left|\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right|\left|\frac{1}{\bar{k}_l'}\left(\frac{2}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2} - C_w\cos\frac{C_w\bar{k}_l'}{2}\right)\right| \tag{37}$$

In other words, $\bar{x}_f$ is mainly used in phase calculations, and its influence on amplitude can be neglected. In the above equations, agreement with the formulae according to Jensen et al. can be confirmed if $\bar{k}_w = 0$ and $C_w = C_b = 1$ are assumed. Although the complexity of the numerical expressions of the proposed formulae is virtually unchanged from that of Jensen's formulae, these are higher-order formulae from the viewpoint that the effects of the ship hull-form parameters C_b and C_w are given proper consideration, and phase information can be clearly obtained by $\bar{x}_f$.

Agreement of the asymptotic value of the proposed formula (34) for heave with the exact equation (12) can be confirmed. However, the asymptotic value of formula (35) for pitch in the long wave length region is as follows:

$$\bar{E}_5^{FK} \sim i\bar{k}_l'\frac{C_w^3}{12} - \bar{x}_f C_w \qquad \text{as } k \to 0 \tag{38}$$

Comparing the right side of (38) with the exact equation (13), it can be understood that the quantity which is equivalent to the nondimensional restoring force coefficient of pitch, $dC_b/L^2 \times \mathrm{GM_L}$, corresponds to the expression shown in (39).

$$\frac{dC_b}{L^2}\mathrm{GM_L} \leftrightarrow C_b^{-0.15}\frac{C_w^3}{12} \tag{39}$$

The right side of (39) is an expression which was obtained by multiplying the nondimensional restoring force coefficient of pitch $C_w^3/12$ when $\bar{B}_w(\bar{x})$ is approximated by a rectangular distribution (right side of Eq. (30)) by the correction factor $C_b^{-0.15}$. In spite of the fact that this expression is different from the left side of Eq. (39), it is not a poor approximation, and as can be confirmed from Fig. 5, its accuracy presents no problems for practical application in the long wave length region. Although we also studied approximation of $\bar{B}'(\bar{x})$ by a trapezoidal distribution, rather than by a rectangular distribution as in Eq. (30), there was no large improvement in accuracy that would justify the increased complexity of the formula. As a result, formulae (34) and (35) in which an approximation by a rectangular distribution was corrected by (33), were adopted as the proposed formulae in this research, as these formulae provide both simplicity and accuracy.

3.4 Sway and Yaw

When the transverse section geometry of the hull is considered as a rectangle with breadth $B'(\bar{x})$ and depth $d'(\bar{x})$, the section Froude-Krylov force in the *y* direction $\bar{f}_2^{FK}(\bar{x})$ is as follows:

$$\bar{f}_2^{FK}(\bar{x}) = i\{1 - e^{-kd'(\bar{x})}\}\left\{\frac{2}{KB}\sin\frac{\bar{k}_w \bar{B}'(\bar{x})}{2}\right\}e^{-i\bar{k}_l\bar{x}} \tag{40}$$

By substituting Eq. (40) into Eq. (7), performing an approximation of Eq. (24) and removing the Smith correction factor from the integrant by using the equivalent draft d_e, $\bar{E}_2^{FK}$ and $\bar{E}_6^{FK}$ are expressed as shown below.

$$\bar{E}_2^{FK} \cong i(1 - e^{-kd_e})\left(\frac{2}{KB}\sin\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F} e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} \tag{41}$$

$$\bar{E}_6^{FK} \cong i(1 - e^{-kd_e})\left(\frac{2}{KB}\sin\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F} \bar{x}e^{-i\bar{k}_l\bar{x}}\bar{B}'(\bar{x})\mathrm{d}\bar{x} \tag{42}$$

Although the definitional equation of $\bar{f}_2^{FK}(\bar{x})$ shown in Eq. (6) is based on the surface integral, it can be replaced by the surface integral on the transverse section of φ_0 by applying the Gauss gradient theorem to that equation. Based on this fact, the equivalent draft is approximated as $d_e = dC_{vp}$, considering the influence of thinness below the waterline. However, for $\bar{E}_6^{FK}$, this is given as $d_e = dC^2_{vp}$ considering the asymptotic value in the long wave length region, as described at the end of this section. The breadth $\bar{B}'(\bar{x})$ is assumed to be a rectangular distribution of the area C_w with LCB as its center:

$$\bar{B}'(\bar{x}) = \begin{cases} 1 & \text{for} \quad |\bar{x}| \leq C_w/2 \\ 0 & \text{otherwise} \end{cases} \tag{43}$$

Finally, the following simplified formulae are obtained by substituting Eq. (43) into Eqs. (41) and (42).

$$\bar{E}_2^{FK} = i(1 - e^{-kdC_{vp}})\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}\right) \tag{44}$$

$$\bar{E}_6^{FK} = \left(1 - e^{-kdC_{vp}^2}\right)\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\frac{1}{\bar{k}_l}\left(\frac{2}{\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2} - C_w\cos\frac{C_w\bar{k}_l}{2}\right) \tag{45}$$

The comparison of the results by the developed formulae shown as Eq. (44) and Eq. (45) with the values obtained by the numerical calculations are shown in Fig. 6 and Fig. 7, respectively. It can be understood that the proposed formulae have satisfactory accuracy for all ship types and wave conditions. $\Re[\bar{E}_2^{FK}]$ and $\Im[\bar{E}_6^{FK}]$ are 0 by the proposed formulae and can also be considered as substantially 0 in the numerical calculations, as the calculated values were at most about $\Re[\bar{E}_2^{FK}] = 0.01$ and $\Im[\bar{E}_6^{FK}] = 0.002$. This means that it is appropriate to consider the center of action of Froude-Krylov force in the *y* direction is at the LCB. The proposed formulae consider the hull-form to be box-shaped with the dimensions $LC_w \times B \times dC_{vp}$, which is the same as for $\bar{E}_3^{FK}$ and $\bar{E}_5^{FK}$. Section 3.3 explained that the accuracy of the *z* direction Froude-Krylov force decreased in the short wave length region of longitudinal waves if the hull form is considered as box-shaped. On the contrary, because the force in the *y* direction in longitudinal waves is inherently 0, the formulae for $\bar{E}_2^{FK}$ and $\bar{E}_6^{FK}$ possess sufficient accuracy for practical application even without the correction like Eq. (33).

If the simplified formula for sway shown as Eq. (44) is expanded by *k*, agreement of its asymptotic value in the long wave length range with the exact value given by Eq. (14) can be confirmed. However, when the simplified formula for yaw in Eq. (45) is expanded to the second order of *k*, it is expressed as follows:

$$\bar{E}_6^{FK} \sim \bar{k}_w \bar{k}_l \frac{d}{B} \frac{C_w C_b^2}{12} \qquad \text{as} \quad k \to 0 \tag{46}$$

Comparing the right sides of the above Eq. (46) and the exact equation Eq. (15), the following correspondence can be observed:

$$\frac{1}{L^3 B d} \int_{V_H} (x^2 - y^2) \mathrm{d}V \;\leftrightarrow\; \frac{C_w C_b^2}{12} \tag{47}$$

The equivalent draft was assumed to be $d_e = dC_{vp}^2$ for $\bar{E}_6^{FK}$ as a result of considering the correspondence shown in Eq. (47). That is, the right side of Eq. (47) derived by assuming $d_e = dC_{vp}^2$ is a good approximation of the integral of the left side. If the equivalent draft d_e is assumed to be $d_e = dC_{vp}$, i.e., the same as for $\bar{E}_2^{FK}$, the right side of (47) becomes $C_w^2 C_b/12$, and its approximation accuracy will decrease. In fact, it was found that the overall estimation accuracy of $\bar{E}_6^{FK}$ when the equivalent draft was assumed to be dC_{vp}^2 was higher than when dC_{vp} was assumed.

3.5 Roll

First, the section Froude-Krylov moment around the x-axis (waterline height) and the Froude-Krylov moment are written as $\bar{f}_{4O}^{FK}(\bar{x})$ and $\bar{E}_{4O}^{FK}$, respectively, and from the definitional equation (5) of n_4, the relationship of the values $\bar{f}_4^{FK}(\bar{x})$ and $\bar{E}_4^{FK}$ around the center of gravity is as follows:

$$\begin{aligned} \bar{f}_4^{FK}(\bar{x}) &= \bar{f}_{4O}^{FK}(\bar{x}) + \bar{z}_G \bar{f}_2^{FK}(\bar{x}) \\ \bar{E}_4^{FK} &= \bar{E}_{4O}^{FK} + \bar{z}_G \bar{E}_2^{FK} \end{aligned} \tag{48}$$

In the following, the moments $\bar{f}_{4O}^{FK}(\bar{x})$ and $\bar{E}_{4O}^{FK}$ around the x-axis will be considered.

When the section shape is considered as a rectangle with breadth $B'(\bar{x})$ and depth $d'(\bar{x})$, the section Froude-Krylov moment $\bar{f}_{4O}^{FK}(\bar{x})$ is as shown by the following equation.

$$\begin{aligned} \bar{f}_{4O}^{FK}(\bar{x}) = {} & i e^{-i\bar{k}_l \bar{x}} \left\{ \frac{2}{KB} \sin \frac{\bar{k}_w \bar{B}'(\bar{x})}{2} \right\} \left[\frac{1 - \{1 + kd'(\bar{x})\} e^{-kd'(\bar{x})}}{KB} \right] \\ & - i e^{-i\bar{k}_l \bar{x} - kd'(\bar{x})} \frac{1}{\bar{k}_w} \left\{ \frac{2}{\bar{k}_w} \sin \frac{\bar{k}_w \bar{B}'(\bar{x})}{2} - \bar{B}'(\bar{x}) \cos \frac{\bar{k}_w \bar{B}'(\bar{x})}{2} \right\} \end{aligned} \tag{49}$$

The first term on the right side is the contribution from the left and right side walls, and the second term is the contribution from the bottom surface. Although Eq. (49) is the similar to the simplified estimation formula for the effective wave slope coefficient proposed by Umeda et al. [11)]. However, in the estimation method for E_4^{FK} according to Umeda et al., the information for $d'(x)$ and $B'(x)$ is given for each transverse section, and numerical integration of $f_4^{FK}(x)$ is required. Thus, while the estimation accuracy of the coefficient proposed by Umeda et al. is high, the number of parameters considered necessary is also correspondingly large.

Here, Eq. (49) is substituted into Eq. (7), the approximation shown as Eq. (24) is applied to the first term on the right side of Eq. (49) and the following approximation is applied to the second term.

$$\frac{2}{\bar{k}_w} \sin \frac{\bar{k}_w \bar{B}'}{2} - \bar{B}' \cos \frac{\bar{k}_w \bar{B}'}{2} \cong \bar{B}'^3 \left(\frac{2}{\bar{k}_w} \sin \frac{\bar{k}_w}{2} - \cos \frac{\bar{k}_w}{2} \right) \tag{50}$$

The approximation in Eq. (50) is based on the fact that the leading term when the left side is expanded by $\bar{k}_w$ is proportional to $\bar{B}'^3$. Furthermore, because the first and second terms on the right side of (49) are integrals related to n_y and n_z, respectively, different shape approximations should be performed by the two. These are distinguished by using the different equivalent drafts d_{e1} and d_{e2} and breadths $\bar{B}_1'(\bar{x})$ and $\bar{B}_2'(\bar{x})$, respectively. Based on the above, $\bar{E}_{4O}^{FK}$ is expressed as shown in Eq. (51):

$$\bar{E}_{40}^{FK} \cong i\left\{\frac{1-(1+kd_{e1})e^{-kd_{e1}}}{kB}\right\}\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F}\bar{B}_1'(\bar{x})e^{-i\bar{k}_l\bar{x}}\mathrm{d}\bar{x} \\ - ie^{-kd_{e2}}\frac{1}{\bar{k}_w}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}-\cos\frac{\bar{k}_w}{2}\right)\int_{\bar{x}_A}^{\bar{x}_F}\{\bar{B}_2'(\bar{x})\}^3 e^{-i\bar{k}_l\bar{x}}\mathrm{d}\bar{x} \tag{51}$$

Continuing, let us consider the approximate value of the integrals of the right side of Eq. (51). Because the first term on the right side of Eq. (51), i.e., the term associated with the side walls, is strong influenced by the draft as the lever of the moment, let $d_{e1} = d$. Assuming that $\bar{B}_1'(\bar{x})$ is a rectangle with an area C_b centered on LCB:

$$\bar{B}_1'(\bar{x}) = \begin{cases} 1 & \text{for} \quad |\bar{x}| \le C_b/2 \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

LCB was assumed as the center because it was inferred that this term is the same as $\bar{E}_2^{FK}$ and $\bar{E}_6^{FK}$, as it is an integral related to n_y, and the area was assumed to be C_b so that the integral value of $d_{e1}B_1'(x)$ is identical to the displacement volume. Under these assumptions, the first term on the right side of Eq. (51) can be expressed as follows:

$$(\text{First term}) = i\left\{\frac{1-(1+kd)e^{-kd}}{kB}\right\}\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l}\sin\frac{C_b\bar{k}_l}{2}\right) \tag{53}$$

Next, the second term on the right side of Eq. (51), that is, the term associated with the bottom surface, is an integral related to n_z. Therefore, as in the Froude-Krylov forces of heave and pitch, $d_{e2} = dC_{vp}$, and $\bar{B}_2'(\bar{x})$ is considered to be equivalent to the waterline breadth $\bar{B}_w'(\bar{x})$ and is considered as having a trapezoidal distribution with an area of C_w centered on LCF:

$$\bar{B}_2'(\bar{x}) = \begin{cases} 1 & \text{for} \quad |\bar{x}-\bar{x}_f| \le C_w - 0.5 \\ \dfrac{0.5-|\bar{x}|}{1-C_w} & \text{for} \quad C_w - 0.5 < |\bar{x}-\bar{x}_f| \le 0.5 \end{cases} \tag{54}$$

If the geometry of Eq. (54) is adopted, the expression of the integral value $\{\bar{B}_2'(\bar{x})\}^3 e^{-i\bar{k}_l\bar{x}}$ will be complex. Therefore, simplification is performed without reducing estimation accuracy in beam sea using the fact that the ship longitudinal distribution of the incident wave front expressed by $e^{-i\bar{k}_l\bar{x}}$ is $e^{-i\bar{k}_l\bar{x}} = 1$ in a beam sea. That is, in a beam sea, the integral value of $\{\bar{B}_2'(\bar{x})\}^3 e^{-i\bar{k}_l\bar{x}}$ using Eq. (54) with a trapezoidal distribution can be expressed simply, as follows:

$$\int_{\bar{x}_A}^{\bar{x}_F}\{\bar{B}_2'(\bar{x})\}^3\mathrm{d}\bar{x} = \frac{3C_w - 1}{2} \tag{55}$$

Based on this fact, if the distribution of $\bar{B}_2'(\bar{x})$ can be considered as a "rectangular distribution with an area of $(3C_w - 1)/2$ centered on LCF," complexification of the integral can be avoided while maintaining accuracy in beam seas. In this case, the integral in the second term on the right side of Eq. (51) can be expressed as shown in Eq. (56).

$$\int_{\bar{x}_A}^{\bar{x}_F}\{\bar{B}_2'(\bar{x})\}^3 e^{-i\bar{k}_l\bar{x}}\mathrm{d}\bar{x} = e^{-i\bar{k}_l\bar{x}_f}\frac{2}{\bar{k}_l}\sin\frac{(3C_w-1)\bar{k}_l}{4} \tag{56}$$

As a result, the second term on the right side of Eq. (51) becomes the following:

$$(\text{Second term}) = -ie^{-i\bar{k}_l\bar{x}_f - kdC_{vp}}\frac{1}{\bar{k}_w}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}-\cos\frac{\bar{k}_w}{2}\right)\left\{\frac{2}{\bar{k}_l}\sin\frac{(3C_w-1)\bar{k}_l}{4}\right\} \tag{57}$$

Finally, the following simplified formula was obtained as the Froude-Krylov moment of roll around the center of gravity:

$$\begin{aligned}\bar{E}_4^{FK} = i\left\{\frac{1-(1+kd)e^{-kd}}{kB}\right\}\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l}\sin\frac{C_b\bar{k}_l}{2}\right) \\ -ie^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\frac{1}{\bar{k}_w}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}-\cos\frac{\bar{k}_w}{2}\right)\left\{\frac{2}{\bar{k}_l}\sin\frac{(3C_w-1)\bar{k}_l}{4}\right\} \quad + \bar{z}_G\bar{E}_2^{FK}\end{aligned} \tag{58}$$

Figure 8 shows the comparison with the numerical calculation values for the developed simplified formula shown in Eq. (58). Although a slight reduction in accuracy can be seen in the short wave length region, it can be understood that this formula has sufficient practical accuracy as a simplified formula.

When the terms on the right side of the proposed formula in Eq. (58) are expanded by k, the results for the respective terms asymptotically approach the following values:

$$(\text{First term}) \sim -i\bar{k}_w\frac{dC_b}{B^2}\left\{-\frac{d}{2}\right\} \qquad \text{as} \quad k\to 0 \tag{59}$$

$$(\text{Second term}) \sim \ -i\bar{k}_w\frac{dC_b}{B^2}\left\{\frac{B^2}{dC_b}\frac{(3C_w-1)}{24}\right\} \qquad \text{as} \quad k\to 0 \tag{60}$$

$$(\text{Third term}) \sim -i\bar{k}_w\frac{dC_b}{B^2}\{-z_G\} \qquad \text{as} \quad k\to 0 \tag{61}$$

When compared with the exact value given by Eq. (16), it can be seen that the sum of the contents enclosed in the curly brackets in Eqs. (59) to (61) is in agreement with GM, and in order from the top, these contents correspond to z_B, BM and $-z_G$ on the right side of Eq. (20). Because shape of the side walls is approximated as box-shape in the first item (i.e., Eq. (59)), $z_B = -d/2$. In the second item (Eq. (60)), from Eqs. (20) and (55), the waterline breadth $\bar{B}_w(\bar{x})$ becomes BM when approximated by a trapezoidal distribution. In formulation of the second term, if the distribution of $\bar{B}_2'(\bar{x})$ is simply approximated by a rectangle having area C_w, the result will diverge from the actual value of BM, resulting in a decrease in accuracy in the long wave length region. Accompanying this, a decrease in accuracy in the short wave length region was also confirmed. Although the right side of Eq. (55) was used in the area of the distribution of $\bar{B}_2'(\bar{x})$ to maintain accuracy in beam seas, this also leads to improved estimation accuracy under all wave conditions.

3.6 Formulae of Pitch/Roll Moment Using Longitudinal/Transverse Metacentric Height

As explained previously, Eq. (35) for the pitch moment $\bar{E}_5^{FK}$ shown in the earlier section 3.3 and Eq. (58) for the roll moment $\bar{E}_4^{FK}$ in section 3.5 asymptotically approach values approximating the restoring force coefficient, which is the exact asymptotic value in the long wave length region. In contrast, if it is acceptable to use the restoring force coefficient of a ship, that is, the longitudinal/transverse metacentric heights, in the formula, formulae for $\bar{E}_5^{FK}$ and $\bar{E}_4^{FK}$ which take the exact asymptotic values can be expressed. Therefore, this section describes the expression of $\bar{E}_5^{FK}$ and $\bar{E}_4^{FK}$ using the longitudinal and transverse metacentric heights, and compares the results with those of formulae shown in Eq. (35) and Eq. (58), which were already developed.

First, let us consider the equation for $\bar{E}_5^{FK}$. In formula in Eq. (35) for $\bar{E}_5^{FK}$, the $C_w\bar{k}_l'$ dependent functions are transformed as shown below:

$$\frac{1}{\bar{k}_l'}\left\{\frac{2}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}-C_w\cos\frac{C_w\bar{k}_l'}{2}\right\} = \bar{k}_l\underline{\frac{C_w^3C_b^{-0.15}}{12}}\left\{3\left(\frac{2}{C_w\bar{k}_l'}\right)^2\left(\frac{2}{\bar{k}_l'C_w}\sin\frac{C_w\bar{k}_l'}{2}-\cos\frac{C_w\bar{k}_l'}{2}\right)\right\} \tag{62}$$

From Eq. (39), the underlined portion on the right side is a quantity which corresponds to the nondimensional restoring force coefficient of pitch $dC_b/L^2\times\mathrm{GM_L}$. Therefore, the following expression of $\bar{E}_5^{FK}$ can be obtained by replacing this with $dC_b/L^2\times\mathrm{GM_L}$:

$$\bar{E}_5^{FK}=e^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left\{i\bar{k}_l\frac{dC_b}{L^2}\mathrm{GM_L}f\left(C_w\bar{k}_l'\right)-\frac{2\bar{x}_f}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}\right\} \tag{63}$$

Here, $f(x)$ is the following function, which asymptotically approaches 1 as $x\to 0$.

$$f(x)=\frac{12}{x^2}\left(\frac{2}{x}\sin\frac{x}{2}-\cos\frac{x}{2}\right)=1+O(x^2) \tag{64}$$

When the underlined portion in Eq. (62) is replaced with $dC_b/L^2\times\mathrm{GM_L}$, Eq. (63) asymptotically approaches the exact value in Eq. (13) in the long wave length region, assuming the correct value of $\mathrm{GM_L}$ is used. Moreover, if $\bar{x}_f$ in Eq. (63) is neglected, it is possible to obtain a composition with an easily-understood physical meaning expressed by the product of the Smith correction factor, the wave length dependent function $f\left(C_w\bar{k}_l'\right)$, which approaches 1 in the long wave length region, and the correct asymptotic value $\bar{k}_l dC_b/L^2\times\mathrm{GM_L}$. A slight improvement in accuracy was confirmed with Eq. (63) in comparison with Eq. (35), not only in the long wave length region, but also in the short wave length region. Accordingly, use of Eq. (63) is recommended in cases where the longitudinal metacentric height $\mathrm{GM_L}$ is known.

Next, let us consider the formula for $\bar{E}_4^{FK}$. In formula shown in Eq. (58) for $\bar{E}_4^{FK}$, when the formula is simplified on the precondition of $\beta=\pi/2$, that is, in beam sea, and k is taken to the second term by a Maclaurin expansion, $\bar{E}_4^{FK}$ is expressed as follows:

$$\bar{E}_4^{FK}=-\frac{ikdC_b}{B}\left\{\underline{-\frac{d}{2}}\left(1-\frac{2}{3}kd\right)\underline{-z_G}\left(1-kdC_{vp}\right)+\underline{\frac{B^2}{dC_b}\frac{3C_w-1}{24}}\left(1-kdC_{vp}\right)\right\}+O(k^3)\cong -ikBe^{-kdC_{vp}}\frac{dC_b}{B^2}\mathrm{GM} \tag{65}$$

Approximation on the extreme right side of Eq. (65) is a result which considers the correspondence of the sum of the underlined portion of Eq. (65) to GM, as explained in section 3.5, and $e^{-kdC_{vp}}\sim 1-kdC_{vp}$. Although this equation was simplified by limiting its application to beam seas, in order to treat oblique waves, the following Eq. (66) was obtained by replacing kB in Eq. (65) with $\bar{k}_w$ based on the correspondence with Eq. (16), and then multiplying by a correction factor by the longitudinal nondimensional wave number $(2/C_w\bar{k}_l)\sin(C_w\bar{k}_l/2)$ (value when the water plane is approximated as a rectangle with dimensions of $C_wL\times B$).

$$\bar{E}_4^{FK}=-i\bar{k}_w e^{-kdC_{vp}}\left(\frac{2}{C_w\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}\right)\frac{dC_b}{B^2}\mathrm{GM} \tag{66}$$

The equation is very simple in comparison with Eq. (58), in which GM is not used, and is also an extremely clear equation in physical terms, as it is the product of the restoring force coefficient $dC_b/B^2\times\mathrm{GM}$ and the wave slope of the sub-surface $\bar{k}_w e^{-kdC_{vp}}$. The accuracy of Eq. (66) when GM is known decreases slightly from that of Eq. (58) (Fig. 8) in the wave length range shorter than $\lambda/L=0.7$, as shown in Fig. 9, but nevertheless is generally satisfactory. Furthermore, unlike Eq. (58), phase information cannot be obtained with Eq. (66), as its real part is 0. However, in comparison with Eq. (58), the asymptotic value of Eq. (66) in the long wave length region is exact, and Eq. (66) is also superior from the viewpoints of simplicity and a composition consisting of easy-to-understand physical quantities. Moreover, since the transverse metacentric height GM is a very basic quantity and is also known in many cases, Eq. (66) is considered to be amply practical as a simplified formula.

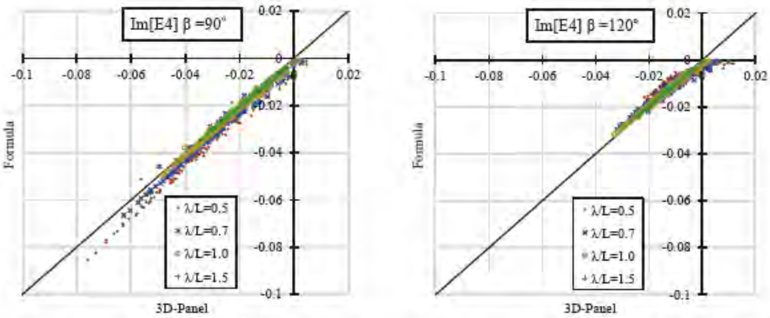


Figure 9 Comparison of $\bar{E}_4^{FK}$ between proposed Formula (66) and numerical calculation.

3.7 Points to Note in Calculations

The preceding sections have presented simplified formulae for 6 degrees of freedom. However, in calculating these simplified formulae, it is necessary to pay attention to the handling of conditions under which the denominator becomes 0. Analytically, there is no problem in taking finite limit values, but in numerical calculations, excess numerical errors or rounding errors may occur in longitudinal waves for which $\bar{k}_w$ becomes 0, and transverse waves for which $\bar{k}_l$ becomes 0, resulting in unreasonable values. In such cases, it can be avoided by the method of assigning limit values by condition branching, or normal values can be obtained more simply, by shifting the wave direction very slightly (by about 0.1°) from 0° or 90°.

Although the developed formulae are expressed by the complex amplitude, when obtaining the amplitude, it is only necessary to take the absolute values assuming $\bar{x}_f = 0$, as described in section 3.3. Where the phase is concerned, it is sufficient to calculate the argument $\arg(E_i^{FK})$ of the complex amplitude as shown in Eq. (1), but when using a solver that cannot handle complex numbers, such numbers must be divided into the real part and imaginary part. In this case, complexification of the equation can be avoided by approximation as shown in Eq. (67) and neglecting the second and higher terms of $\bar{x}_f$.

$$e^{-i\bar{k}_l\bar{x}_f} \cong 1 - i\bar{k}_l\bar{x}_f \tag{67}$$

The formula values shown in Fig. 3 to Fig. 8 were also calculated in that manner.

Care is also necessary when using the proposed formulae, the instant when the crest of a wave reaches the center of gravity position of a ship is used as a time reference. If the instant when the wave crest reaches the position $x = x_1$ is to be used as the time reference, the proposed formula E_i^{FK} should be multiplied by the phase as shown below.

$$E_i^{FK} \rightarrow e^{i\bar{k}_l\bar{x}_1}E_i^{FK} \tag{68}$$

4. CONCLUSIONS

In this paper, simplified formulae for the Froude-Krylov forces of 6 degrees of freedom (6-DOF), which are applicable without limitation of the ship type and size, were developed to enable simple estimation of ship motion in waves. The development of practical formulae of the Froude-Krylov force which can estimated only by several main dimensions of ship based on a physical discussion in this research is a new attempt without precedent in the past. The authors believe that we have succeeded in developing accurate, generally applicable formulae which are sufficient for practical application. A summary of the developed formulae and hull-form approximations is presented in Table 1.

The key points in the development of the simplified formulae are as follows.

a) In order to develop formulae that are applicable without limitation as to the ship type, an approach was adopted in which the Froude-Krylov force is expressed by an elementary function using the ship's main parameters and wave conditions as variables, by approximating the ship hull-form by functions that are uniquely determined by 8 main parameters of the ship (L, B, d, C_b, C_w, C_m, x_f (= LCF - LCG) and KG). For the pitch and roll moments, formulae supposing cases in which the longitudinal metacentric height GM_L and the transverse metacentric height GM are known were also proposed.

b) In approximation of the ship hull-form, the hull-form is approximated by an appropriate shape for each of the 6 DOFs based on geometrical considerations and the shape that results in a correct asymptotic value in the long wave length region.

It is known that the asymptotic value of the Froude-Krylov force in the long wave length region corresponds to the restoring force coefficient. The asymptotic values of the proposed formulae approach the exact values for surge, sway and heave forces, and approach the values of the restoring force coefficient which is approximated by the main parameters for the moments of roll, pitch and yaw. Furthermore, the above-mentioned formulae for the moments of pitch and roll using the longitudinal or transverse metacentric heights approach the exact values.

c) Appropriate consideration is given to the phase difference with respect to the incident wave by assuming LCB as ship center position in terms related to n_x and n_y, and assuming LCF as ship center position in terms related to n_z.

Finally, in concluding this paper, the features and evaluation of the developed simplified formulae and the results produced thereby may be summarized as follows.

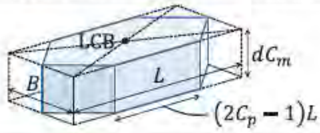
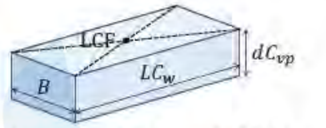
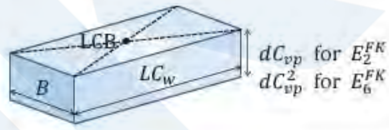
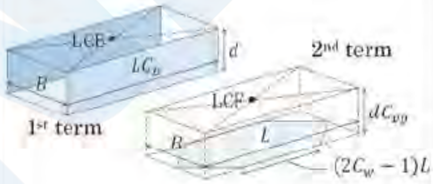
i) The proposed formulae have high estimation accuracy for 77 actual ships under two different loading conditions (full load, ballast), without limitation as to the ship type or size, under all wave direction and wave length conditions. In particular, the accuracy of the formulae increases in the longer wave length region. Because the Froude-Krylov force does not depend on the ship speed when it is based on a uniform flow approximation, these formulae can be applied to substantially all wave conditions within the range of linear theory.

ii) Because the necessary requirements for calculations are limited to only 8 main parameters of the ship (9 in case the longitudinal/transverse metacentric heights are used), rational estimation of the Froude-Krylov force is possible even without detailed information concerning the ship's hull. This is particularly useful in evaluation of ship motility in the initial stage of design. Among the main parameters, x_f is not a general main parameter and is more difficult to obtain than the other items, but since it is a parameter that mainly influences the phase, information on x_f is not necessary when the aim is to investigate amplitude.

iii) To the best of the authors' knowledge, there are no past examples in which estimation formulae for the Froude-Krylov force expressed only by the main parameters of a ship were obtained by a theoretical approach. The research by Jensen et al. [3] for a similar purpose presented formulae for the Froude-Krylov force for a box-shaped ship, and corrected the formulae by using a fineness coefficient. In contrast, in the formulae proposed here, the complexity of the numerical expressions is essentially unchanged from those proposed by Jensen et al., but the proposed formulae are sophisticated formulae in that the influence of the ship's hull-form parameters is considered appropriately based on a geometrical consideration, and phase information can be clearly obtained. Although simplified estimation formulae which are used in stability standards exist for the Froude-Krylov moment of roll [11], information on the geometry of each transverse section of the hull is required. In contrast, reasonable estimation is possible by the proposed formulae using only the main parameters.

iv) As mentioned in the Introduction, the simplified formulae for the Froude-Krylov force have an especially high value for simplicity in estimating roll and surge. Since these are also main components among the hydrodynamic forces for other modes of motion, it is expected that the formulae developed in this research can be used effectively in simple estimations of ship motion in waves. For example, because a dominant parameter that does not exist in the motion and acceleration provisions of CSR was discovered by proposed formulae, it is expected that use of the formulae will lead to improvement of the accuracy and general applicability of the formulae.

Table 1 Summary of proposed formulae and hull-form approximations.

Mode	Proposed Formula	Hull form approximation
Surge	$\bar{E}_1^{FK} = i(1-e^{-kdC_m})\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{kL}\sin\frac{C_p\bar{k}_l}{2}\right)\left\{\frac{2}{(1-C_p)\bar{k}_l}\sin\frac{(1-C_p)\bar{k}_l}{2}\right\}$	
Heave	$\bar{E}_3^{FK} = e^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}\right)$	 + Correction for $\bar{k}_l \to C_b^{-0.15}\bar{k}_l$
Pitch around COG	$\bar{E}_5^{FK} = ie^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\frac{1}{\bar{k}_l'}\left\{\left(\frac{2}{\bar{k}_l'}+2i\bar{x}_f\right)\sin\frac{C_w\bar{k}_l'}{2}-C_w\cos\frac{C_w\bar{k}_l'}{2}\right\}$ $\bar{E}_5^{FK} = ie^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}\right)\left\{\bar{k}_l\frac{dC_b}{L^2}\mathrm{GM_L}f(C_w\bar{k}_l')+\frac{2i\bar{x}_f}{\bar{k}_l'}\sin\frac{C_w\bar{k}_l'}{2}\right\}$ where, $f(x)=\frac{12}{x^2}\left(\frac{2}{x}\sin\frac{x}{2}-\cos\frac{x}{2}\right)$	
Sway	$\bar{E}_2^{FK} = i(1-e^{-kdC_{vp}})\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}\right)$	
Yaw around COG	$\bar{E}_6^{FK} = (1-e^{-kdC_{vp}^2})\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\frac{1}{\bar{k}_l}\left(\frac{2}{\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}-C_w\cos\frac{C_w\bar{k}_l}{2}\right)$	
Roll around COG	$\bar{E}_4^{FK} = i\left\{\frac{1-(1+kd)e^{-kd}}{kB}\right\}\left(\frac{2}{kB}\sin\frac{\bar{k}_w}{2}\right)\left(\frac{2}{\bar{k}_l}\sin\frac{C_b\bar{k}_l}{2}\right)$ $-ie^{-i\bar{k}_l\bar{x}_f-kdC_{vp}}\frac{1}{\bar{k}_w}\left(\frac{2}{\bar{k}_w}\sin\frac{\bar{k}_w}{2}-\cos\frac{\bar{k}_w}{2}\right)\left\{\frac{2}{\bar{k}_l}\sin\frac{(3C_w-1)\bar{k}_l}{4}\right\}$ $+\bar{z}_G\bar{E}_2^{FK}$ $\bar{E}_4^{FK} = -i\bar{k}_w e^{-kdC_{vp}}\left(\frac{2}{C_w\bar{k}_l}\sin\frac{C_w\bar{k}_l}{2}\right)\frac{dC_b}{B^2}\mathrm{GM}$	
Where:	$C_p=\frac{C_b}{C_m}$, $C_{vp}=\frac{C_b}{C_w}$, $\bar{k}_l=kL\cos\beta$, $\bar{k}_w=kB\sin\beta$, $\bar{k}_l'=C_b^{-0.15}\bar{k}_l$, $\bar{x}_f=\frac{\mathrm{LCF}-\mathrm{LCG}}{L}$, $\bar{z}_G=\frac{\mathrm{KG}-d}{B}$	

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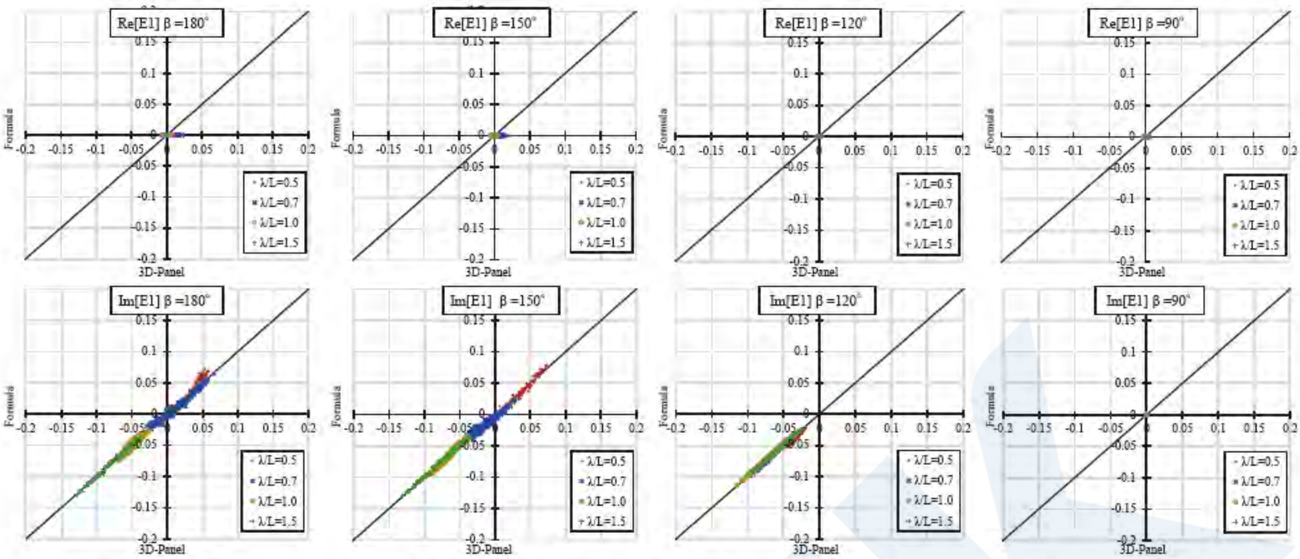

Figure 3 Comparison of $\bar{E}_1^{FK}$ between proposed formula and numerical calculation for target ships.

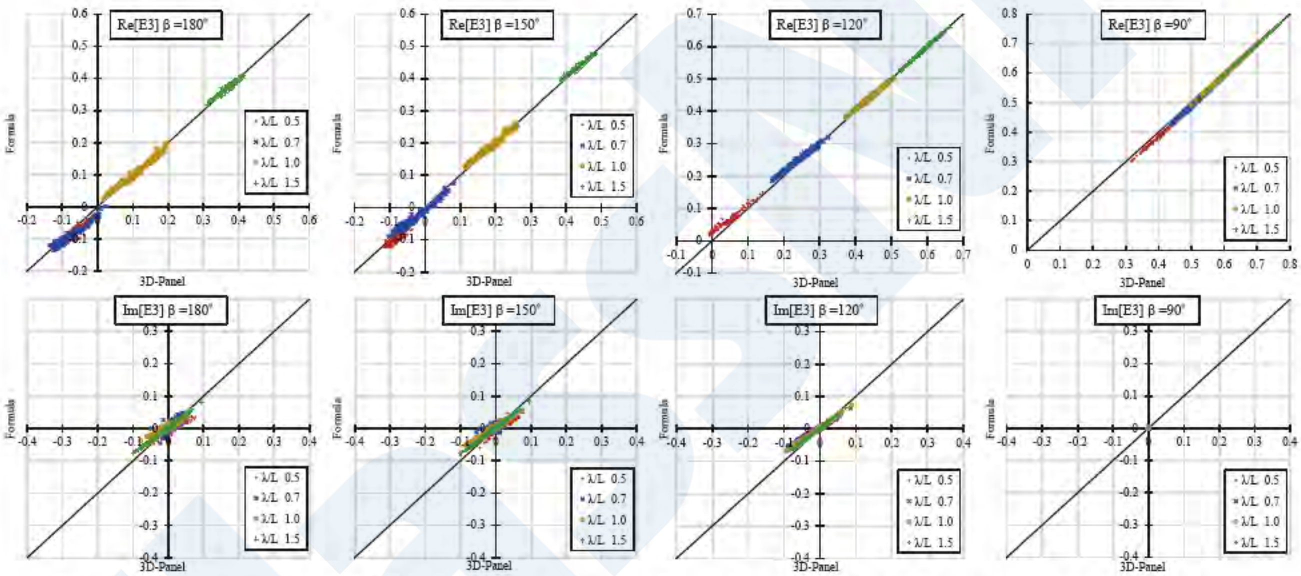

Figure 4 Comparison of $\bar{E}_3^{FK}$ between proposed formula and numerical calculation for target ships.

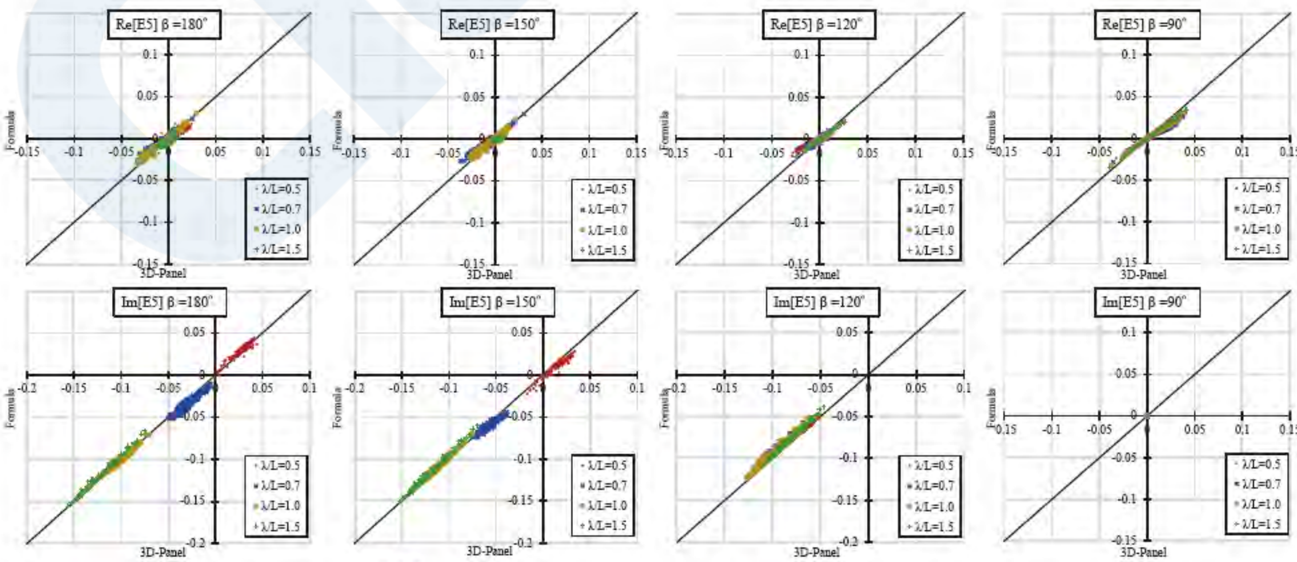

Figure 5 Comparison of $\bar{E}_5^{FK}$ between proposed formula and numerical calculation for target ships.

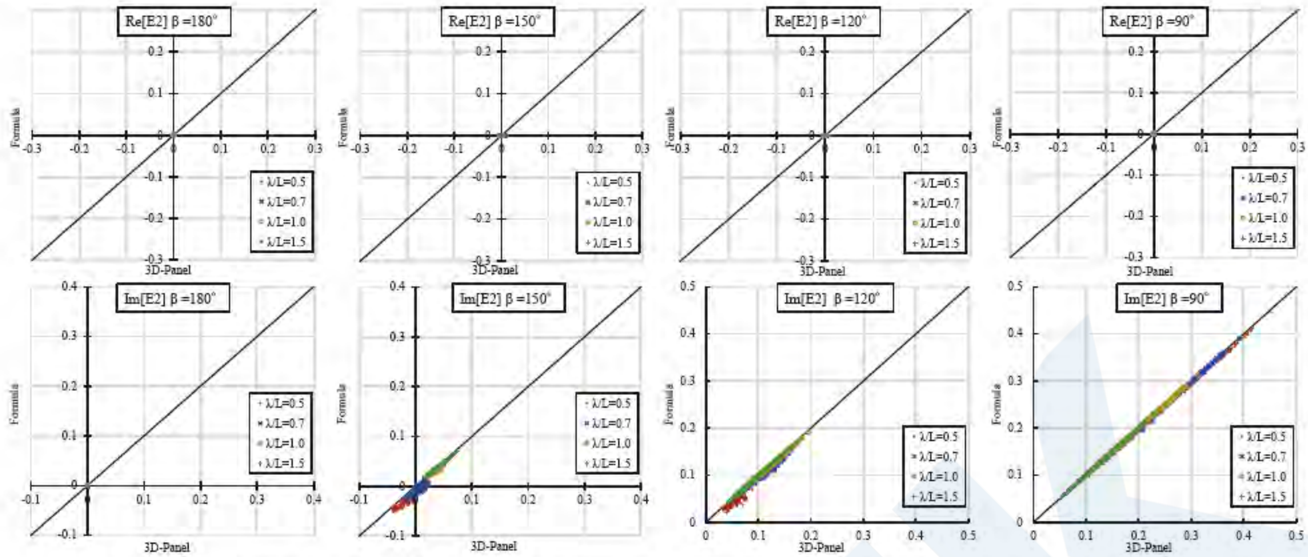

Figure 6 Comparison of $\bar{E}_2^{FK}$ between proposed formula and numerical calculation for target ships.

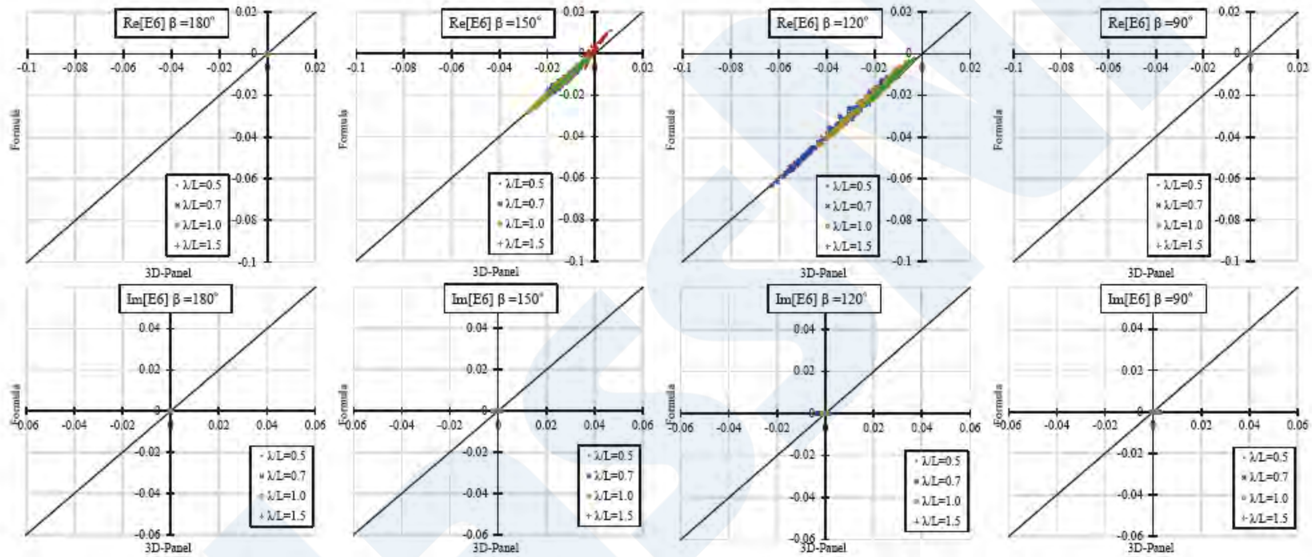

Figure 7 Comparison of $\bar{E}_6^{FK}$ between proposed formula and numerical calculation for target ships.

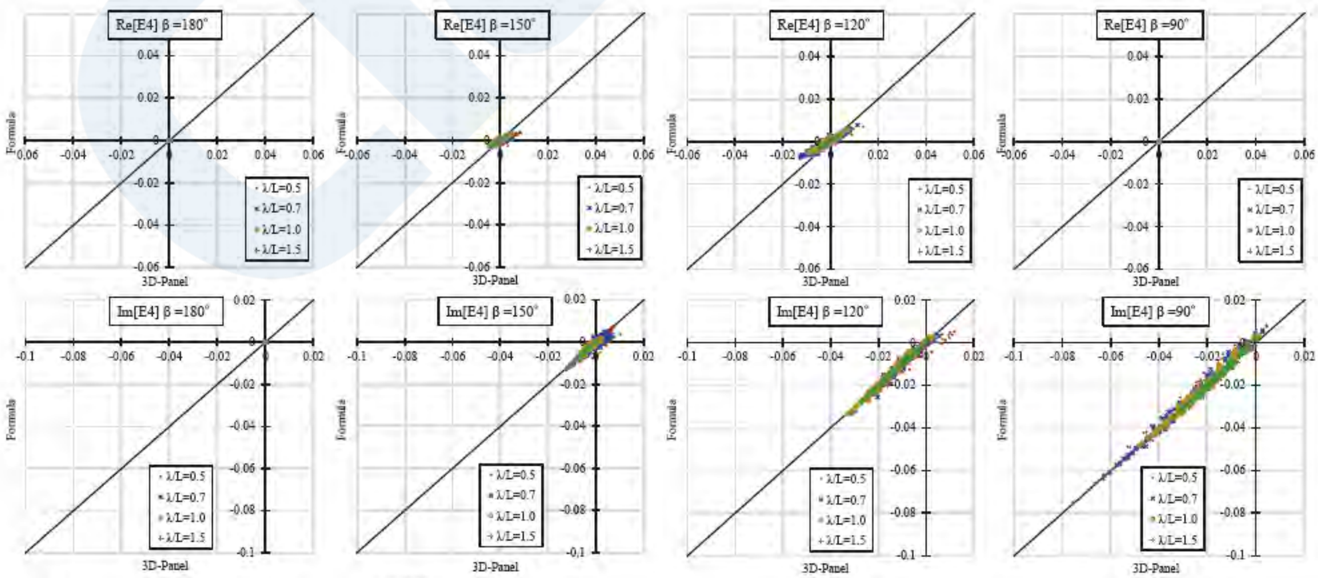

Figure 8 Comparison of $\bar{E}_4^{FK}$ between proposed formula and numerical calculation for target ships.